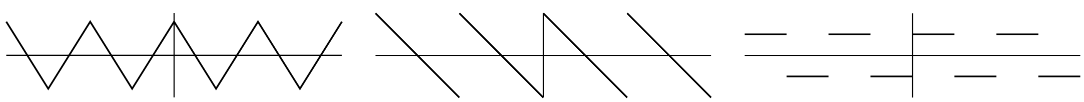
Fourier Series

A **Fourier series** is an infinite series of the form

$$a + \sum_{n=1}^{\infty} b_n \cos(n\omega x) + \sum_{n=1}^{\infty} c_n \sin(n\omega x).$$

Virtually any periodic function that arises in applications can be represented as the sum of a Fourier series. For example, consider the three functions whose graph are shown below:



These are known, respectively, as the **triangle wave** $\mathsf{\Lambda}(x)$, the **sawtooth wave** $\mathsf{N}(x)$, and the **square wave** $\sqcap\!\sqcup(x)$. Each of these functions can be expressed as the sum of a Fourier series:

$$\begin{aligned}
\mathsf{\Lambda}(x) &= \cos x + \frac{\cos 3x}{3^2} + \frac{\cos 5x}{5^2} + \frac{\cos 7x}{7^2} + \frac{\cos 9x}{9^2} + \cdots \\
\mathsf{N}(x) &= \sin x + \frac{\sin 2x}{2} + \frac{\sin 3x}{3} + \frac{\sin 4x}{4} + \frac{\sin 5x}{5} + \cdots \\
\sqcap\!\sqcup(x) &= \sin x + \frac{\sin 3x}{3} + \frac{\sin 5x}{5} + \frac{\sin 7x}{7} + \frac{\sin 9x}{9} + \cdots
\end{aligned}$$

Fourier series are critically important to the study of differential equations, and they have many applications throughout the sciences. In addition, Fourier series played an important historical role in the development of analysis, and the desire to prove theorems about their convergence was a large part of the motivation for the development of Lebesgue integration.

These notes develop Fourier series on the level of calculus. We will not be worrying about convergence, and we will not be not be proving that any given function is

n	$\cos nx$	$\sin nx$
2	$\cos^2 x - \sin^2 x$	$2\cos x \sin x$
3	$\cos^3 x - 3\cos x \sin^2 x$	$3\cos^2 x \sin x - \sin^3 x$
4	$\cos^4 x - 6\cos^2 x \sin^2 x + \sin^4 x$	$4\cos^3 x \sin x - 4\cos x \sin^3 x$
5	$\cos^5 x - 10\cos^3 x \sin^2 x + 5\cos x \sin^4 x$	$5\cos^4 x \sin x - 10\cos^2 x \sin^3 x + \sin^5 x$

Table 1: Multiple-angle formulas.

actually equal to the sum of its Fourier series. We will revisit the theoretical aspects of this topic later in the course after we have defined the Lebesgue integral and proven Lebesgue's dominated convergence theorem.

Trigonometric Polynomials

A **trigonometric polynomial** is a polynomial expression involving $\cos x$ and $\sin x$:

$$\cos^5 x + 6\cos^3 x \sin^2 x + 3\sin^4 x + 2\cos^2 x + 5$$

Because of the identity $\cos^2 x + \sin^2 x = 1$, most trigonometric polynomials can be written in several different ways. For example, the above polynomial can be rewritten as

$$5\cos^3 x \sin^2 x + 3\sin^4 x + \cos^3 x - 2\sin^2 x + 7$$

Fourier Sums

A **Fourier sum** is a Fourier series with finitely many terms:

$$5 \,+\, 3\sin 2x \,+\, 4\cos 5x \,-\, 3\sin 5x \,+\, 2\cos 8x.$$

Every Fourier sum is actually a trigonometric polynomial, and any trigonometric polynomial can be expressed as a Fourier sum.

Converting a Fourier sum to a trigonometric polynomial is fairly straightforward: simply substitute the appropriate multiple-angle identity for each $\cos nx$ and $\sin nx$ (see Table 1).

It is less obvious that every trigonometric polynomial can be expressed as a Fourier sum. This depends on the three **product-to-sum formulas**:

$$\begin{aligned}\cos A \cos B &= \frac{1}{2}\cos(A-B) + \frac{1}{2}\cos(A+B) \\ \sin A \cos B &= \frac{1}{2}\sin(A-B) + \frac{1}{2}\sin(A+B) \\ \sin A \sin B &= \frac{1}{2}\cos(A-B) - \frac{1}{2}\cos(A+B).\end{aligned}$$

These identities allow us to transform any product of trigonometric functions into a sum. By applying them repeatedly, we can remove all of the multiplications from a trigonometric polynomial, resulting in a Fourier sum.

Alternatively, one can use these identities to derive **power-reduction formulas** for $\cos^j x \sin^k x$, the first few of which are listed below:

	$j=0$	$j=1$	$j=2$	$j=3$
$k=0$	1	$\cos x$	$\dfrac{1+\cos 2x}{2}$	$\dfrac{3\cos x+\cos 3x}{4}$
$k=1$	$\sin x$	$\dfrac{\sin 2x}{2}$	$\dfrac{\sin x+\sin 3x}{4}$	$\dfrac{2\sin 2x+\sin 4x}{8}$
$k=2$	$\dfrac{1-\cos 2x}{2}$	$\dfrac{\cos x-\cos 3x}{4}$	$\dfrac{1-\cos 4x}{8}$	$\dfrac{2\cos x-\cos 3x-\cos 5x}{16}$
$k=3$	$\dfrac{3\sin x-\sin 3x}{4}$	$\dfrac{2\sin 2x-\sin 4x}{8}$	$\dfrac{2\sin x+\sin 3x-\sin 5x}{16}$	$\dfrac{3\sin 2x-\sin 6x}{32}$

These formulas tell us how to convert each term of a trigonometric polynomial directly into a Fourier sum.

Orthogonality

There is a nice integral formula for finding the coefficients of any Fourier sum. This is based on the **orthogonality** of the functions $\cos nx$ and $\sin nx$:

Theorem 1 Orthogonality Relations

If $j, k \in \mathbb{N}$, then:

$$\int_{-\pi}^{\pi} \cos jx \cos kx \, dx \;=\; \int_{-\pi}^{\pi} \sin jx \sin kx \, dx \;=\; \begin{cases} \pi & \text{if } j = k \\ 0 & \text{otherwise,} \end{cases}$$

and

$$\int_{-\pi}^{\pi} \sin jx \cos kx \, dx \;=\; 0.$$

PROOF For $n \in \mathbb{Z}$, observe that

$$\int_{-\pi}^{\pi} \sin nx \, dx \;=\; 0 \qquad \text{and} \qquad \int_{-\pi}^{\pi} \cos nx \, dx \;=\; \begin{cases} 2\pi & \text{if } n = 0 \\ 0 & \text{otherwise.} \end{cases}$$

If $j, k \in \mathbb{N}$, we can use the product-to-sum identities to deduce that

$$\int_{-\pi}^{\pi} \cos jx \cos kx \, dx \;=\; \int_{-\pi}^{\pi} \frac{\cos (j-k)x + \cos (j+k)x}{2} \, dx \;=\; \begin{cases} \pi & \text{if } j = k \\ 0 & \text{otherwise,} \end{cases}$$

and

$$\int_{-\pi}^{\pi} \sin jx \sin kx \, dx \;=\; \int_{-\pi}^{\pi} \frac{\cos (j-k)x - \cos (j+k)x}{2} \, dx \;=\; \begin{cases} \pi & \text{if } j = k \\ 0 & \text{otherwise,} \end{cases}$$

and

$$\int_{-\pi}^{\pi} \sin jx \cos kx \, dx \;=\; \int_{-\pi}^{\pi} \frac{\sin (j-k)x + \sin (j+k)x}{2} \, dx \;=\; 0. \qquad \blacksquare$$

In general, the **inner product** of two functions f and g on an interval $[a, b]$ is

$$\langle f, g \rangle \;=\; \int_a^b f(x)\, g(x) \, dx.$$

A collection $\mathcal{F}$ of nonzero functions on $[a, b]$ is said to be **orthogonal** if $\langle f, g \rangle = 0$ for all $f, g \in \mathcal{F}$ with $f \neq g$. According to the above theorem, the functions

$$\{\cos nx\}_{n \in \mathbb{N}} \cup \{\sin nx\}_{n \in \mathbb{N}}$$

are orthogonal on the interval $[-\pi, \pi]$. Note that these functions are also orthogonal to the constant function 1.

This definition of orthogonality is related to the notion of orthogonality in linear algebra. Specifically, let $C([a,b])$ be the vector space of all real-valued continuous functions on the interval $[a,b]$. Then the formula for $\langle f, g\rangle$ given above defines an inner product on this vector space (analogous to the dot product on $\mathbb{R}^n$), under which orthogonal functions are the same as orthogonal vectors.

Theorem 2 Fourier Coefficients

Let

$$f(x) \;=\; a + \sum_{n=1}^{N} b_n \cos nx + \sum_{n=1}^{N} c_n \sin nx.$$

Then

$$a \;=\; \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\, dx.$$

Furthermore, for all $n \in \{1, \ldots, N\}$,

$$b_n \;=\; \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\, dx \qquad \textit{and} \qquad c_n \;=\; \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\, dx.$$

PROOF The formula for a is fairly obvious. To derive the formula for the b's, observe that

$$\int_{-\pi}^{\pi} f(x)\cos kx\, dx$$
$$= a\int_{-\pi}^{\pi} \cos kx\, dx + \sum_{n=1}^{N} b_n \int_{-\pi}^{\pi} \cos nx \cos kx\, dx + \sum_{n=1}^{N} c_n \int_{-\pi}^{\pi} \sin nx \cos kx\, dx.$$

Applying the orthogonality relations reduces this to

$$\int_{-\pi}^{\pi} f(x)\cos kx\, dx \;=\; b_k \pi$$

and the formula for b_k follows. The derivation of the formula for c_k is similar. ■

The formulas in the theorem above can be written as follows:

$$a \;=\; \frac{\langle f, 1\rangle}{\langle 1, 1\rangle}, \qquad b_n \;=\; \frac{\langle f, \cos nx\rangle}{\langle \cos nx, \cos nx\rangle} \qquad \text{and} \qquad c_n \;=\; \frac{\langle f, \sin nx\rangle}{\langle \sin nx, \sin nx\rangle}.$$

From the point of view of linear algebra, these are special cases of a formula that holds for any collection of orthogonal vectors. Specifically, let $\mathbf{u}_1, \ldots, \mathbf{u}_n$ be orthogonal vectors in an inner product space, and let

$$\mathbf{v} \;=\; \lambda_1 \mathbf{u}_1 + \cdots + \lambda_n \mathbf{u}_n.$$

Then

$$\lambda_k \;=\; \frac{\mathbf{v} \cdot \mathbf{u}_k}{\mathbf{u}_k \cdot \mathbf{u}_k}$$

for each k, where $\cdot$ denotes the inner product of vectors.

Corollary 3 Uniqueness of Fourier Sums

Let

$$f(x) \;=\; a \,+\, \sum_{n=1}^{N} b_n \cos nx \,+\, \sum_{n=1}^{N} c_n \sin nx$$

and

$$g(x) \;=\; A \,+\, \sum_{n=1}^{N} B_n \cos nx \,+\, \sum_{n=1}^{N} C_n \sin nx.$$

Then $f = g$ if and only if $a = A$ and $b_n = B_n$ and $c_n = C_n$ for all n.

Finally, we should mention the following famous formula for the inner product of two trigonometric polynomials. This follows directly from the orthogonality relations:

Theorem 4 Inner Product Formula

Let

$$f(x) \;=\; a \,+\, \sum_{n=1}^{N} b_n \cos nx \,+\, \sum_{n=1}^{N} c_n \sin nx$$

and

$$g(x) \;=\; A \,+\, \sum_{n=1}^{N} B_n \cos nx \,+\, \sum_{n=1}^{N} C_n \sin nx.$$

Then

$$\int_{-\pi}^{\pi} f(x)\, g(x)\, dx \;=\; 2\pi a A \,+\, \sum_{n=1}^{N} \pi b_n B_n \,+\, \sum_{n=1}^{N} \pi c_n C_n.$$

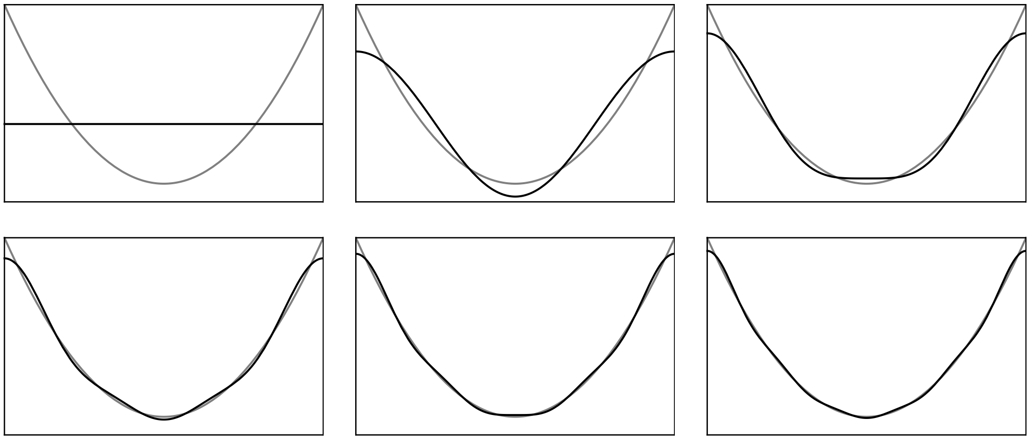


Figure 1: Six partial sums of the Fourier series for x^2.

Fourier Series

We have seen how the coefficients of the Fourier sum for a trigonometric polynomial $f(x)$ can be found using definite integrals. The same formulas can be used to define Fourier coefficients for any function f:

Definition: Fourier Coefficients for *f*
The **Fourier coefficients** for a function $f\colon [-\pi, \pi] \to \mathbb{R}$ are the real number a and the sequences b_n and c_n defined by the following formulas:

$$a = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\,dx, \quad b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,\cos nx\,dx, \quad c_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,\sin nx\,dx.$$

Definition: Fourier Series for *f*
The **Fourier series** for a function $f\colon [-\pi, \pi] \to \mathbb{R}$ is the sum

$$a + \sum_{n=1}^{\infty} b_n \cos nx + \sum_{n=1}^{\infty} c_n \sin nx.$$

where a, b_n, and c_n are the Fourier coefficients for f.

If f is a trigonometric polynomial, then its corresponding Fourier series is finite, and the sum of the series is equal to $f(x)$. The surprise is that the Fourier series usually converges to $f(x)$ even if f isn't a trigonometric polynomial.

EXAMPLE 1 Let $f\colon [-\pi,\pi] \to \mathbb{R}$ be the function $f(x) = x^2$. The integrals

$$a \;=\; \frac{1}{2\pi}\int_{-\pi}^{\pi} x^2\,dx, \quad b_n \;=\; \frac{1}{\pi}\int_{-\pi}^{\pi} x^2\cos nx\,dx, \quad c_n \;=\; \frac{1}{\pi}\int_{-\pi}^{\pi} x^2\sin nx\,dx$$

yield the following Fourier coefficients:

$$a \;=\; \frac{\pi^2}{3}, \qquad b_n \;=\; (-1)^n\frac{4}{n^2}, \qquad \text{and} \qquad c_n \;=\; 0.$$

Thus the Fourier series for f is

$$\frac{\pi^2}{3} + \sum_{n=1}^{\infty}(-1)^n\frac{4}{n^2}\cos nx \;=\; \frac{\pi^2}{3} - 4\cos x + \frac{4\cos 2x}{2^2} + - \frac{4\cos 3x}{3^2} + \cdots .$$

This series converges uniformly to $f(x)$ on the interval $[-\pi,\pi]$. Figure 1 shows the first six partial sums of this series, together with the parabola $y = x^2$.

Of course, the series only converges to x^2 on the interval $[-\pi,\pi]$. Over the real line, the sum of the series is periodic with period 2π, as shown in Figure 2. ■

EXAMPLE 2 Now consider the function $f\colon [-\pi,\pi] \to \mathbb{R}$ defined by $f(x) = x$. The Fourier coefficients for this function are

$$a \;=\; 0, \qquad b_n \;=\; 0, \qquad \text{and} \qquad c_n \;=\; (-1)^{n+1}\frac{2}{n},$$

so the Fourier series for f is

$$\sum_{n=1}^{\infty}(-1)^{n+1}\frac{2}{n}\sin nx \;=\; 2\sin x - \frac{2}{2}\sin 2x + \frac{2}{3}\sin 3x - \frac{2}{4}\sin 4x + \cdots .$$

Note that the coefficients of this series are essentially the harmonic series, which diverges. Thus, it is very unclear that this series will even converge for a typical value of x.

The first eight partial sums of this series are shown in Figure 3. As you can see, this series appears to converge to $f(x)$ for most values of x. Indeed, it does converge

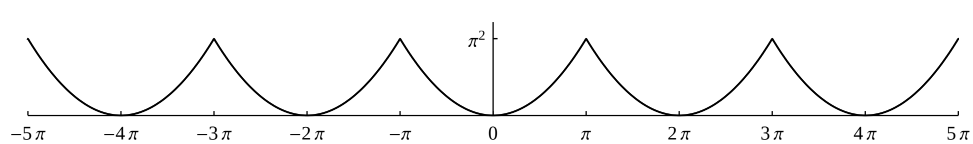


Figure 2: The sum of the Fourier series for x^2.

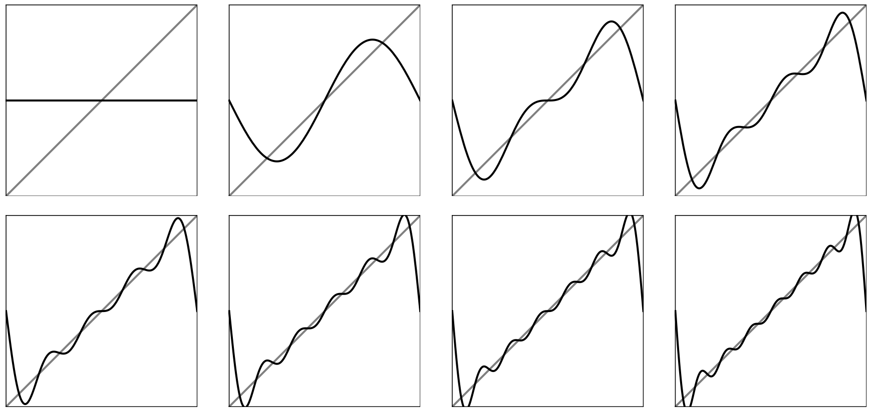


Figure 3: Eight partial sums of the Fourier series for x.

to $f(x)$ for all values of x in the interval $(-\pi, \pi)$, though this is relatively difficult to prove.

Also, as you can see from the graphs, all of the partial sums of the Fourier series have roots at $-\pi$ and π. It follows that the sum of the series also has roots at these points. Therefore, the Fourier series for $f(x)$ converges pointwise to the function

$$g(x) \;=\; \begin{cases} x & \text{if } -\pi < x < \pi \\ 0 & \text{if } x = \pm\pi. \end{cases}$$

on the interval $[-\pi, \pi]$.

Figure 4 shows the sum of this Fourier series over the real line. This function is similar to the "sawtooth wave" discussed in the introduction, although the graph in Figure 4 is more explicit about the behavior at the discontinuities. ■

As these examples show, the issue of which functions can be represented as Fourier series is a bit complicated. As a general rule, if $f\colon [-\pi, \pi] \to \mathbb{R}$ is any reasonably well-behaved function, then the Fourier series for f converges to $f(x)$ for "almost all" values of x. Unfortunately, it is rather difficult to prove any general results

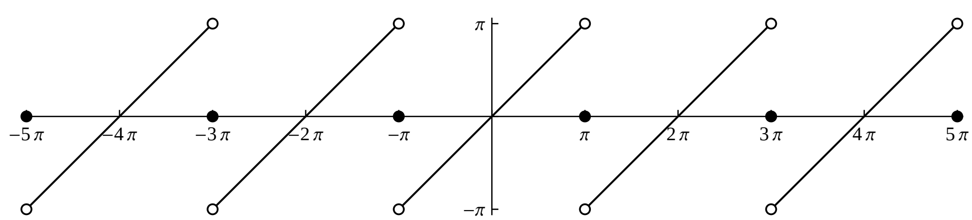


Figure 4: The sum of the Fourier series for x.

about convergence of Fourier series without the help of measure theory and Lebesgue integration.

Applications

Now that we know how to find the Fourier series for a typical function, we would like to discuss how these series are used in mathematics and physics. Fourier series were first developed to help in the solution of certain very important partial differential equations that arise in the study of physical systems.

We will be considering the following three partial differential equations. The first involves a function $f(x_1, \ldots, x_n)$ with no time-dependence, while the other two involve a function $f(x_1, \ldots, x_n, t)$:

$$\nabla^2 f = 0, \qquad \nabla^2 f = \frac{\partial f}{\partial t}, \qquad \text{and} \qquad \nabla^2 f = \frac{\partial^2 f}{\partial t^2}$$

These equations[1] are respectively known as **Laplace's equation**, the **heat equation**, and the **wave equation**. These are arguably the three most important partial differential equations, and much of the study of PDE's is devoted to understanding just these three. As we shall see, Fourier series can be quite helpful for solving them.

Laplace's Equation

Let D^2 be the unit disc centered at the origin in $\mathbb{R}^2$, and consider Laplace's equation for a function $f\colon D^2 \to \mathbb{R}$. This has the form:

$$\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = 0.$$

In general, solutions to Laplace's equation are called **harmonic functions**.

In the same way that an ordinary differential equation requires initial conditions to specify a unique solution, a partial differential equation requires **boundary conditions**. In the present case, what we need to know is the value of the function on the boundary circle:

[1]The expression $\nabla^2 f$ in these equations is called the **Laplacian**, and is defined by the formula

$$\nabla^2 f = \frac{\partial^2 f}{\partial x_1^2} + \cdots + \frac{\partial^2 f}{\partial x_n^2}.$$

Intuitively, the Laplacian measures how much the value of f at point differs from the average value of f at nearby points.

> **Dirichlet's Problem**
> Given a function $g\colon S^1 \to \mathbb{R}$, find a harmonic function $f\colon D^2 \to \mathbb{R}$ that agrees with g on S^1.

In principle, Dirichlet's problem should have a solution for virtually any function g. The reason for this is based on physics, specifically the distribution of heat inside solid bodies. According to the theory of heat transfer, the temperature $T(x, y)$ inside any solid body in a steady state must be a harmonic function. Imagine, then, that we heat and cool the edge of a metal disc so as to maintain a specific temperature function on the boundary circle. If we wait a long time, the temperature function $T(x, y)$ inside the disc should reach a steady state, and this will be the desired solution to Dirichlet's problem.

So how can we solve Dirichlet's problem mathematically? Well, consider the functions $\varphi_n\colon D^2 \to \mathbb{R}$ and $\psi_n\colon D^2 \to \mathbb{R}$ defined by

$$\varphi_n(x, y) \;=\; r^n \cos n\theta \qquad \text{and} \qquad \psi_n(x, y) \;=\; r^n \sin n\theta,$$

where (r, θ) are the polar coordinates for a point (x, y). It is not difficult to show that φ_n and ψ_n are harmonic functions for any $n \in \mathbb{N}$. Moreover, any expression such as

$$3 \,+\, 2r^2\cos 2\theta \,-\, 5r^3\sin 3\theta \,+\, r^8\sin 8\theta$$

is also a harmonic function.[2] This gives us a large number of different solutions to work with.

In particular, observe that $\varphi_n(x, y)$ and $\psi_n(x, y)$ restrict to the functions $\cos n\theta$ and $\sin n\theta$ on the unit circle. Therefore, we now know how to solve Dirichlet's problem whenever $g(\theta)$ is a trigonometric polynomial.

EXAMPLE 3 Let $g\colon S^1 \to \mathbb{R}$ be the function

$$g(\theta) \;=\; 5\cos\theta + 4\sin 2\theta - 6\cos 5\theta.$$

Find a harmonic function $f\colon D^2 \to \mathbb{R}$ that agrees with g on S^1.

SOLUTION The desired harmonic function is

$$f(x, y) \;=\; 5r\cos\theta + 4r^2\sin 2\theta - 6r^5\cos 5\theta. \qquad \blacksquare$$

How does this help in general? Well, given any function $g\colon S^1 \to \mathbb{R}$, we can try to express g as a Fourier series involving $\cos n\theta$ and $\sin n\theta$:

$$g(\theta) \;=\; a \,+\, \sum_{n=1}^{\infty} b_n \cos n\theta \,+\, \sum_{n=1}^{\infty} c_n \sin n\theta.$$

[2] In general, any linear combination of solutions to Laplace's equation is again a solution. A differential equation with this property is called **linear**.

Assuming this series actually converges to g, we can treat the Fourier series as though it were a trigonometric polynomial, giving us the following harmonic function:

$$f(x,y) \;=\; a \,+\, \sum_{n=1}^{\infty} b_n r^n \cos n\theta \,+\, \sum_{n=1}^{\infty} c_n r^n \sin n\theta.$$

Assuming all of this works, we should be able to solve Dirichlet's problem for any function g that can be expressed as the sum of a Fourier series.

EXAMPLE 4 Let $g\colon S^1 \to \mathbb{R}$ be the function

$$g(x,y) \;=\; \begin{cases} 1 & \text{if } x > 0 \\ 1/2 & \text{if } x = 0 \\ 0 & \text{if } x < 0 \end{cases}.$$

Find a harmonic function $f\colon D^2 \to \mathbb{R}$ that agrees with g on S^1.

SOLUTION In terms of θ, we have

$$g(\theta) \;=\; \begin{cases} 1 & \text{if } -\pi/2 < \theta < \pi/2 \\ 1/2 & \text{if } \theta = \pm\pi/2 \\ 0 & \text{otherwise.} \end{cases}$$

This function is indeed the sum of a Fourier series. The integrals for the Fourier coefficients are

$$a \;=\; \frac{1}{2\pi}\int_{-\pi/2}^{\pi/2} dx, \qquad b_n \;=\; \frac{1}{\pi}\int_{-\pi/2}^{\pi/2} \cos nx\, dx, \qquad \text{and} \qquad c_n \;=\; \frac{1}{\pi}\int_{-\pi/2}^{\pi/2} \sin nx\, dx.$$

This gives $a = 1/2$ and $c_n = 0$, while b_n is the sequence

$$\frac{2}{\pi},\;\; 0,\;\; -\frac{2}{3\pi},\;\; 0,\;\; \frac{2}{5\pi},\;\; 0,\;\; ,-\frac{2}{7\pi},\;\; \ldots$$

Thus

$$g(\theta) \;=\; \frac{1}{2} \,+\, \frac{2}{\pi}\left(\cos\theta \,-\, \frac{\cos 3\theta}{3} \,+\, \frac{\cos 5\theta}{5} \,-\, \frac{\cos 7\theta}{7} \,+\, \cdots\right).$$

Then

$$f(x,y) \;=\; \frac{1}{2} \,+\, \frac{2}{\pi}\left(r\cos\theta \,-\, \frac{r^3\cos 3\theta}{3} \,+\, \frac{r^5\cos 5\theta}{5} \,-\, \frac{r^7\cos 7\theta}{7} \,+\, \cdots\right)$$

should be the desired harmonic function.

A plot of this function is shown in Figure 5. The gray level indicates the value of the function, and the contours for $0.1, 0.2, \ldots, 0.9$ are shown. ■

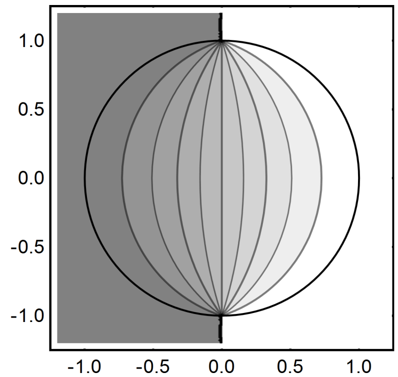


Figure 5: A harmonic function on the unit disc.

The Heat Equation

Consider a long metal rod whose temperature varies with position. Assuming the rod is thermally insulated from its surroundings, heat will slowly diffuse from the hot portions of the rod to the cool portions until the temperature becomes uniform. Ignoring constants such as the thermal conductivity of the metal, this diffusion is governed by the equation

$$\frac{\partial T}{\partial t} = \frac{\partial^2 T}{\partial x^2}$$

where $T(x,t)$ is the temperature of the rod at position x and time t. This PDE is a one-dimensional version of the heat equation.

Now, given the initial distribution of the temperature at $t = 0$, we ought to be able to predict how the temperature of the rod will evolve over time. For simplicity, assume that the rod has a length of 2π, with x restricted to the interval $[-\pi, \pi]$. This gives us the following problem:

Heat Diffusion Problem
Given a function $g\colon [-\pi,\pi] \to \mathbb{R}$, find a solution $T\colon [-\pi,\pi] \times \mathbb{R} \to \mathbb{R}$ to the heat equation such that $T(x,0) = g(x)$ for all $x \in [-\pi,\pi]$.

The solution to this problem is similar to our solution to Dirichlet's problem. First we must solve the heat equation in the case where $g(x) = \cos nx$ or $g(x) = \sin nx$. As you can easily verify, the corresponding solutions are

$$\varphi_n(x,t) = e^{-n^2 t}\cos nx \qquad \text{and} \qquad \psi_n(x,t) = e^{-n^2 t}\sin nx.$$

Again, any combination of these will also be a solution to the heat equation:

$$2 + 5e^{-t}\cos t + 8e^{-9t}\sin 3t - 6e^{-25t}\cos 5t$$

This lets us solve the heat equation whenever $g(x)$ is a trigonometric polynomial.

For an arbitrary function $g(x)$, we can follow the same procedure we did before. First we attempt to express g as a Fourier series:

$$g(x) \;=\; a + \sum_{n=1}^{\infty} b_n \cos nx + \sum_{n=1}^{\infty} c_n \sin nx.$$

Assuming this succeeds, we obtain a candidate solution to the heat equation for which $T(x,0) = g(x)$:

$$T(x,t) \;=\; a + \sum_{n=1}^{\infty} b_n\, e^{-n^2 t} \cos nx + \sum_{n=1}^{\infty} c_n\, e^{-n^2 t} \sin nx.$$

Of course, this is all based on conjecture and hope. We have little evidence that the above series will converge in general. Also, this series will only be a solution to the heat equation if it is valid to take its derivative by separately differentiating each term. We will need to develop a lot of theory if we want to make this rigorous.

The Wave Equation

Consider a stringed instrument, such as a guitar, piano, or harp. Such an instrument consists of several long strings held at high tension, which vibrate and produce sound when they are disturbed. The classical **vibrating string problem** is to model the motion of such a string.

Imagine that the string is stretched out along the x-axis, and is only allowed to vibrate in the vertical direction. At each time t, the vertical displacement of the string will be a function $u(x)$, with the shape of the string being the graph of this function. Thus we can describe the motion of the string by a function $u(x,t)$, where x is horizontal position and t is time.

Under ideal conditions, the motion of the string will be governed by the following equation:

$$\frac{\partial^2 u}{\partial x^2} \;=\; \frac{\partial^2 u}{\partial t^2}.$$

This is the one-dimensional version of the wave equation.

In principle, we should be able to solve this equation if we are given the initial shape $u(x,0)$ and initial velocity $(\partial u/\partial t)(x,0)$ of the string. For simplicity, we assume that the string has length π, with x restricted to the interval $[0,\pi]$. In addition, we assume that the string is pinned at its ends, so that $u(0,t) = u(\pi,t) = 0$ for all $t \in \mathbb{R}$:

Vibrating String Problem
Given two functions $g, h\colon [0,\pi] \to \mathbb{R}$, find a solution $u\colon [0,\pi]\times\mathbb{R} \to \mathbb{R}$ to the wave equation satisfying the following conditions:

$$u(0,t) = u(\pi,t) = 0, \qquad u(x,0) = g(x), \qquad \frac{\partial u}{\partial t}(x,0) = h(x).$$

The solution here is slightly more complicated than the previous cases. Consider the following solutions to the wave equation:

$$\psi_n(x,t) \;=\; \sin nx \cos nt \qquad \text{and} \qquad \Psi_n(x,t) \;=\; \sin nx \sin nt.$$

Note that $\psi_n(x,t)$ and $\Psi_n(x,t)$ describe the same sort of vibration, but are temporally out of phase. Each solution $\psi_n(x,t)$ has initial position $g(x) = \sin nx$, but has zero initial velocity. On the other hand, each solution $\Psi_n(x,t)$ restricts to $g(x) = 0$, but has initial velocity $h(x) = n \sin nx$.

These solutions let us solve the wave equation as long as $g(x)$ and $h(x)$ are trigonometric polynomials.[3] Specifically, suppose that

$$g(x) \;=\; \sum_{n=1}^{N} A_n \sin nx \qquad \text{and} \qquad h(x) \;=\; \sum_{n=1}^{N} B_n \sin nx.$$

Then the corresponding solution to the wave equation is

$$u(x,t) \;=\; \sum_{n=1}^{N} A_n \sin nx \cos nt \;+\; \sum_{n=1}^{N} \frac{B_n}{n} \sin nx \sin nt.$$

Note that the factor of $1/n$ cancels with the n that we get from the derivative of $\sin nt$.

More generally, given any functions g and h, we can attempt to express both as Fourier sine series:

$$g(x) \;=\; \sum_{n=1}^{\infty} A_n \sin nx \qquad \text{and} \qquad h(x) \;=\; \sum_{n=1}^{\infty} B_n \sin nx.$$

Then the corresponding solution to the wave equation ought to be

$$u(x,t) \;=\; \sum_{n=1}^{\infty} A_n \sin nx \cos nt \;+\; \sum_{n=1}^{\infty} \frac{B_n}{n} \sin nx \sin nt.$$

Incidentally, the basic solutions $\psi_n(x,t)$ and $\Psi_n(x,t)$ to the wave equation are known as **normal modes**. Physically, these correspond to certain standing waves in the string (see Figure 6). If the string used to produce music, the primary modes ψ_1

[3] Since the string is fixed at both ends, we know that $g(0) = g(\pi) = 0$. Therefore, if g is a trigonometric polynomial, it must be a sum of sines. The same goes for h.

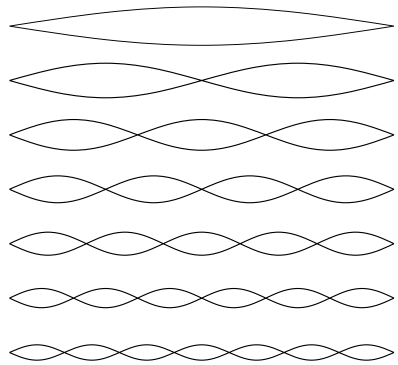


Figure 6: The normal modes $\psi_1, \ldots, \psi_7$.

and Ψ_1 sound the main note, while the remaining modes $\psi_2, \psi_3, \ldots$ and $\Psi_2, \Psi_3, \ldots$ produce certain higher notes known as **harmonics** or **overtones**. The strength of these overtones is responsible for the rich sound of stringed instruments.

Exercises

1. Let $\mathsf{\Lambda}\colon \mathbb{R} \to \mathbb{R}$, $\mathsf{N}\colon \mathbb{R} \to \mathbb{R}$, and $\sqcap\!\sqcup\colon \mathbb{R} \to \mathbb{R}$ be the three functions defined by Fourier series in the introduction.

   (a) Determine the range of each of these functions.

   (b) Draw careful graphs of $\mathsf{N}(x)$ and $\sqcap\!\sqcup(x)$, making sure to show the value of each function at its points of discontinuity.

2. Use the product-to-sum formulas and the table of power-reduction formulas to express $\cos^6 x \sin^6 x$ as a Fourier sum.

3. Let $p_1, p_2, p_3\colon [-1,1] \to \mathbb{R}$ be the functions

$$p_1(x) \;=\; 1, \qquad p_2(x) \;=\; x, \qquad \text{and} \qquad p_3(x) \;=\; 3x^2 - 1$$

   (a) Compute $\langle p_i, p_j \rangle$ for all $i, j \in \{1,2,3\}$. Are p_1, p_2, and p_3 orthogonal?

   (b) Let $q\colon [-1,1] \to \mathbb{R}$ be a quadratic polynomial, and suppose that

$$\langle q, p_1 \rangle = 7, \qquad \langle q, p_2 \rangle = 2, \qquad \text{and} \qquad \langle q, p_3 \rangle = 8.$$

   What is q?

4. Let $f\colon [-\pi, \pi] \to \mathbb{R}$ be the function defined by

$$f(x) \;=\; \sum_{n=1}^{\infty} \frac{\sin nx}{2^n} \;=\; \frac{\sin x}{2} + \frac{\sin 2x}{4} + \frac{\sin 3x}{8} + \frac{\sin 4x}{16} + \cdots.$$

a) Evaluate $\displaystyle\int_{-\pi}^{\pi} f(x)\sin 3x\,dx$.

b) Evaluate $\displaystyle\int_{-\pi}^{\pi} f(x)^2\,dx$.

5. Let $f\colon [-\pi,\pi] \to \mathbb{R}$ be the function

$$f(x) = \begin{cases} 1 & \text{if } 0 \leq x \leq \pi/2 \\ 0 & \text{otherwise.} \end{cases}$$

Compute the Fourier series for $f(x)$ by hand.

6. Compute the Fourier series for $\cos(x/2)$ and e^x on the interval $[-\pi,\pi]$. (Feel free to use Wolfram Alpha for the integrals.)

7. Let $g\colon S^1 \to \mathbb{R}$ be the function $g(x,y) = |y|$. Find a formula for the harmonic function $f\colon D^2 \to S^2$ that agrees with g on S^1. (Feel free to use Wolfram Alpha for the integrals.)